AN ALGORITHM FOR PREDICTING THE VISIBILITY OF THE LUNAR CRESCENT

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Abstract

Dr. Ilyas has called for a better theoretical model of the visibility of the lunar crescent so as to allow for the accurate calculation of lunar date lines. The theoretical approach has a number of fundamental advantages over a purely empirical approach. Theory can account for local conditions, while an empirical rule must assume that weather conditions worldwide are identical to those of the locale where the original data is obtained. Empirical rules take no notice of the important effects of lunar distance or libration. Empirical rules cannot be safely extrapolated past the conditions of the original data. No data exists concerning lunar visibility for telescopic observations, airborne observers, or spaceborne observers, so an empirical method is impossible. The best previous theoretical work has been characterized by Dr. Ilyas as "incomplete and erroneous." Indeed, several of the physical assumptions made are incorrect by over a factor of a thousand. I have approached the problem from a modern astrophysical point of view which includes: (1) extensive photometry of the twilight sky, (2) modeling of atmospheric effects, (3) solving the complex shadowing and forward scattering equations for the lunar surface, (4) incorporating refraction, parallax, pupil diameter, and Stiles-Crawford corrections, (5) the sensitivity of the human eye to detecting an extended source in the shape of a crescent that is unevenly illuminated, and (6) a compact lunar and solar ephemeris accurate to better than an arc-minute for times within several centuries of today. From this model, a computer program has been written which predicts lunar visibility. The FORTRAN program has roughly 60 operative lines with no operation more complex than an arctangent, and it has a fast run time. Versions of the program have been written for drawing lunar date lines and for telescopic observation. This model has been compared with 218 observations which I have compiled. There is excellent agreement between my model and the observations (with the exception of observations 73 and 74 of Fotheringham which the observer himself suggests are in error). The size of the uncertainty zone is roughly 24 degrees in longitude. The lunar date lines calculated by this program are in reasonable agreement with the predictions of Dr. Ilyas, provided that the latitudes are within 40 degrees of the equator and a particular set of observing conditions are assumed. It is found that the longitude of the lunar date lines can be moved by typically 90 degrees depending on whether an "average" or an "excellent" site is being considered. It is hoped that the greatly increased accuracy and the recognition of local conditions will help to spur the acceptance of the lunar date line concept.

Introduction

The visibility of the young crescent moon is an important problem for Islamic calendrics. As such, much effort in the last millennium has gone into constructing prediction algorithms. These efforts can generally be divided into two approaches: empirical and theoretical.

The empirical approach entails the collection of a set of actual observations, from which an empirical rule is derived. In modern times, the only data set used is that compiled by Fotheringham.[1] This data set consists of 76 observations made in Athens, Greece, in the last half of the nineteenth century. From these data, a number of criteria have been established, based (for example) on the time of moonset after sunset or the moon's altitude at sunset.

The best previous theoretical work is that of Bruin.[2] Unfortunately, many of his assumptions are grossly incorrect. For example, his assumed lunar surface brightness is many orders of magnitude in error. Also, he assumes that the twilight sky brightness does not depend on position on the sky for the relevant region, whereas the sky brightness varies by a factor of four. It is naive to equate the visibility of an unevenly illuminated crescent with the visibility of a circular disk of 0.3' diameter as he does. Bruin uses the phys-

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[1] J. K. Fotheringham, *Monthly Notices of the Royal Astronomical Society* 70 (1910): 527.

[2] F. Bruin, *Vistas in Astronomy* 21 (1977): 331.

iological data of Siedentopf[3] without corrections for color, exit pupil diameter, or binocular vision. Finally, Bruin takes no account for changing observational conditions.

The empirical approach has the advantage that the rule is firmly grounded in actual data. A disadvantage is that the rule is only applicable for observing conditions similar to those of the original data set. This is like saying that the conditions for visibility in the American Southwest are the same as for New England. The effect of observing conditions can easily shift the lunar date line by over 90 degrees of longitude. The theoretical approach has the disadvantage that the algorithm must be checked with real data before acceptance. However, the versatility of the approach allows the visibility to be predicted for a wide variety of conditions (for which no systematic observations may exist). Such cases include those with telescopic assistance, or where the observer is at high latitudes or in an airplane.

With this background, I have set about constructing an algorithm for predicting lunar visibility by modeling the actual physics and physiology of the situation (see the section "Program"). Then I have collected 218 observations for use in evaluating the algorithm (see the section "Observations").

Program

I have separated the act of viewing the moon into more than a dozen processes. Each process is then quantitatively modeled as exactly as possible. In some cases, the information used to construct the model was obtained from professional journals. In other cases, the formulae were obtained from private communication of unpublished work by professionals. Some of the equations are based on data I have collected at astronomical observatories. Many derivations are required to cast the available equations into useful formats. In all, roughly half the algorithm is from unpublished sources.

To calculate the visibility of the early moon my program employs a number of equations which describe various relevant physical phenomena. These equations include the following:

1. The twilight sky brightness for "standard" conditions as a function of the sun's depression angle, the altitude above the horizon, and the azimuth relative to the sun. The relations are calibrated from extensive photometry at the

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[3] H. Siedentopf, *Astronamiste Nachtrichten* 271 (1940): 193.

Kitt Peak and Cerro Tololo observatories, and with several published studies.

2. The twilight sky brightness under "nonstandard" conditions are related to the brightness under "standard" conditions by a model of atmospheric scattering.
3. The pupil diameter of the eye as a function of sky brightness.
4. A function which varies at a known rate with the age of the observer.
5. The calculation of the geocentric position of the sun and moon to an accuracy of better than an arc-minute for any time within several centuries of the present epoch.
6. The calculation of the topocentric position of the crescent with corrections for the lunar radius and parallax.
7. Finding the apparent position of the crescent after a correction for refraction. The degree of refraction is a function of air temperature, air pressure, and the observer's altitude.
8. Calculation of the sidereal time to an accuracy of better than a second—provided that an accurate longitude is known.
9. Calculation of the altitudes and azimuths of the sun and moon from the information in equations 5 through 8 via standard trigonometric equations.
10. Calculation of the number of airmasses through which a ray of lunar light must pass by an equation which is accurate to the horizon.
11. Determination of the visual extinction coefficient which is the primary source of uncertainty for this program. The extinction can be reasonably estimated for a site given its altitude, humidity, latitude, and date. The basis for this estimation is published and unpublished monthly data for over sixty sites worldwide.
12. The surface brightness of the moon is a complex and rapidly varying function of the angle between the sun and the moon. The calculation involves macro-shadowing, micro-shadowing, and forward-scattering for a distribution of particle sizes.
13. The visibility of an extended light source against some background is a complex problem. The calculations are based on extensive physiological data and the probabilistic model of photon detection. The physiological data are for competent observers of average visual acuity who know the direction and shape of the experimental stimulus.

The program has no "adjustable parameters," since all the equations are based on "first principles." That is to say, I have no empirical correction factors by which to "fudge" the data. Hence, for a given set of observing conditions, only one answer is possible.

The following parameters are needed to run the program:

1. The date of the new moon. This can be expressed as either a Muslim month and year, as a Christian date, or as a Julian date. The latter two input modes need only be accurate to within a week.
2. The observer's latitude and longitude.
3. The altitude of the site.
4. The temperature, pressure, and humidity at the site.
5. The faintest star visible from the site after darkness has fallen.
6. The age of the observer.

These data can be used to estimate the extinction coefficient, or alternatively, the coefficient can be estimated from independent photometric data from a nearby site. Parameters 3 to 6 need only be known approximately, as their effect on the lunar visibility is relatively small (other than through their effect on the estimate of the extinction coefficient). The predominant uncertainty in the result is caused by the uncertainties in the atmospheric clarity. For most conditions, the effects of the uncertainties in the extinction will dominate over the effects of variable visual acuity.

The output of the program is the age of the moon at the time of first visibility for the specified conditions. This can also be expressed as a Julian date or Christian date. My program also computes the arc of vision, arc of light, and the azimuthal separation for comparison with empirical rules. A modification of my program will calculate the position of the lunar date line. A separate modification will calculate the first visibility of the moon when a telescopic aid is used.

The program is written in both Basic and FORTRAN and has roughly sixty operative lines. The most complicated function used is the arctangent. The run time is extremely fast because all the complexity of the program is in the many needed coefficients, instead of in many loops which would slow the execution.

One of the more important results of my program concerns the effect of atmospheric clarity on the position of the lunar date line. For example, the lunar date line will shift by typically 90 degrees when comparing "average" as opposed to "excellent" sites. With the possibility of observing under "poor" conditions, the total uncertainty in the longitude of the lunar date line can easily be 180 degrees. In other words, a prediction algorithm that does not account for observing conditions may only get the correct hemisphere. Of course, most algorithms will do better than this because they are based on nonextreme conditions. But even for nonextreme conditions (say for extinction coefficients between 0.2 and 0.4), the lunar date line will still move by roughly 90 degrees. One conclusion from this is that any algorithm that ignores atmospheric clarity will have a zone of uncertainty at least 90 degrees wide in longitude. Another conclusion is that, after the relative positions of the sun and the moon are considered, the visual extinction coefficient is the most important parameter for predicting visibility.

In his book, Dr. Ilyas[4] has published a set of lunar date lines which can be compared to my program results. In general, I find close agreement between the two algorithms for latitudes within 40 degrees of the equator and for a particular set of observing conditions. This set of observing conditions is just what is to be expected for Athens in the late 1800's. For sites which differ greatly from Athens (e.g., Saudi Arabia, Egypt, Malaysia, the American Southwest, the American East Coast) the predictions of Ilyas will have large systematic shifts in longitude. Fortunately, Ilyas's predictions can be used with confidence from any site if it is more than ninety degrees away from the published lunar date line.

An interesting question is what is the youngest possible visible crescent for a ground-based naked-eye observer? The optimal conditions are the following:

1. the moon is at perigee;
2. the moon is 90 degrees past a node;
3. the observer's latitude is such that the moon stands directly over the sun at the time of best visibility;
4. the observer's longitude is such that the moon's altitude is optimal at the time of best visibility; and
5. the observer has the clearest skies ever known positively to have existed (i.e., an extinction coefficient of 0.10, such as is seen only rarely from Mauna Kea's summit).

Note that just the first two conditions are satisfactorily fulfilled only rarely at new moon (roughly once a decade on average). Even given this rare opportunity, only one restricted area on the earth's surface meets the third and fourth conditions. Of course, the probabilities are low that this restricted area contains one of several sites good enough to satisfy the fifth condition. Then the site must have the rare perfect skies. Finally, a young sharp-eyed observer must be ready and waiting. If all these unlikely conditions are accepted, then my program calculates that a 10-hour old moon (with an angular distance from the sun of 8 degrees) would be marginally visible. It

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[4] M. Ilyas, *Islamic Calendar, Time and Qibla* (Kuala Lumpur: Berita, 1984).

must be stressed that conditions 1-5 occur, perhaps, once a millenium on average. Relaxing conditions a bit (such as might occur once or twice a century), the minimum observable age will be more like 13 hours.

Observations

Any theoretical model must be tested against real data before acceptance. To this end, I have collected 218 actual observations of the visibility or invisibility of the crescent moon. The majority of the reports were made by professional and amateur observers and are available in the published astronomical literature. Of the unpublished reports, the majority are from the "moon watch" organized by Dr. Doggett on April 28, 1987. Seven remaining observations have been personally collected by me. Thirteen of the 218 reports are for telescopic observation, while 29 are for naked-eye results where a pair of binoculars was used to initially sight the moon. Of the 205 naked-eye reports, 48 are negative sightings where the moon was not detected. Nine reports are observations of the old moon in the morning sky. Roughly half the observations are "critical" in the sense that the moon is only just barely visible or invisible. Eight reports are "trivial" in the sense that the moon was visible on the previous night. (This is not accidental, since recent astronomical literature tends to only report the critical cases.) All reports include the date of observation, the latitude, the longitude, and the details of any optical aid.

For a comparison of the model with the observations, some additional input parameters are needed. The site's altitude can be found from an atlas. The mean temperature, pressure, and humidity for the correct time of year can be found in weather tables. The zenith limiting magnitude in almost all cases will be close to 6.0, although the result is not sensitive to the value chosen here. For most cases, the observer's age is known to within a decade, although once again, this input parameter has only a relatively small effect on the model output.

The above parameters are sufficient to allow an estimation of the visual extinction coefficient. However, in many cases, it is possible to make a better estimate based on photometric data from nearby observatories. To aid in the estimation of atmospheric clarity, I have collected year-round observations from over sixty sites around the world. Unfortunately, I have no photometric data from Athens, Greece, where the many observations of Schmidt and Mommsen were obtained. Currently, I can estimate the extinction to roughly 15 percent in winter and 25 percent in summer.

For each actual lunar visibility observation, the appropriate input parameters were used in conjunction with the model to calculate whether the moon would be visible. In each case, a visibility parameter, R, was calculated, where R is the logarithm of the ratio of the moon's surface brightness to the surface brightness required for marginal visibility at the optimum time of the evening. Hence, a positive value of R implies that the moon should be visible, while a negative value implies invisibility.

A total of 27 (out of 218) model predictions are discrepant in the sense that visibility was predicted yet no moon was seen, or vice versa. The existence of these discrepant predictions is of no surprise, since they are a feature of any model, be it empirical or theoretical. The discrepancies are all for cases of near critical visibility with a lack of perfect knowledge of the observing conditions. This uncertainty in marginal cases translates into an uncertainty in the location of the lunar date line. In effect, there is a zone of uncertainty centered around the lunar date line. In the west end of the zone of uncertainty, the probability of spotting the moon is quite high, while in the east, the probability is quite small.

The R values for the discrepant cases are distributed closely around zero, with the average being 0.03. This implies that the model predictions are well centered in the zone of uncertainty. In other words, the model has no systematic errors. This is encouraging because I have no empirical correction factors to account for any systematic errors.

The 27 discrepant cases can also be used to delineate the width of the zone of uncertainty. For each case, the longitude of the lunar date line at the latitude of the observer was calculated for the relevant parameters. Then the distribution of the longitude differences define the width of the uncertainty zone. For the discrepant cases, the longitude differences are roughly distributed as a Gaussian with a one sigma value of 14 degrees. A comparison with the nondiscrepant cases shows that the probability of a discrepancy falls to 10% when the longitude difference is 12 degrees. Hence, the total width of the uncertainty zone is roughly 24 degrees.

Dr. Doggett of the United States Naval Observatory organized a "moon watch" for April 28, 1987 (the day on which Ramadan started). This "moon watch" was designed to allow for an empirical determination of the lunar date line. These results can then be used to choose among the competing prediction algorithms. The first algorithm, a moonset lag criterion from HM Nautical Almanac Office, gave the easternmost longitude of visibility as 86 west. The second algorithm, that presented by Dr. Ilyas in his book, gave the easternmost longitude of visibility as 40 west. The third algorithm, that presented in this paper, predicted the easternmost longitude of visibility as 75 west for the observing conditions which typically prevail along the eastern coast. Dr. Doggett recruited over fifty observers (primarily astronomers, navigators, and planetarium staff) throughout the United States. Many were clouded out, especially in the Northeast, which had an unseasonably late snowstorm. About 25 of the observers had clear skies. The crescent moon was spotted with the naked eye by many observers including those located in Maryland, Michigan, North Carolina, Florida, Texas, and Louisiana. These sightings indicate that the lunar date line is east of 86 west. However, negative visual sightings were made in Georgia, North Carolina, Michigan, Alabama, Maryland, Iowa, and Florida. This indicates that the lunar date line is significantly west of 40 west. The zone of uncertainty, as defined by these observations, is roughly 20 degrees wide in longitude and is centered on a longitude of 79 west. In summary, of the three prediction programs, the "moon watch" results are in excellent agreement with my program.

Predictions

I have calculated the visibility of the moon for five American cities for the new moons occurring between August 1987 and December 1988 (see the accompanying table). Each entry indicates the age of the moon (in hours) at which it should be first sighted. These predictions are intended both for practical use and as a means to test my model against observations.

Conclusions

It is obvious to any observer of lunar crescents that the observing conditions are an important factor in determining whether the moon is spotted. No previous empirical or theoretical work takes account of observing conditions. In this paper, I have discussed a model of lunar visibility (based on the relevant physics and physiology) which does account for the observing conditions. This model has been implemented as a computer program that is both short and fast.

The real test of the program is in how accurately its model predictions agree with the 218 actual observations I have collected. I find that the agreement between observation and theory is excellent, and that the "zone of uncertainty" is 24 degrees of longitude in width. In particular, the program results are more reliable than any other published algorithm.

It is hoped that the greatly increased accuracy and the recognition of local conditions will help to spur the acceptance of the lunar date line concept.

Age of the Moon at Time of First Visibility

New Moon		Boston	Wash. D.C.	Detroit	Tuscon	Seattle
1987	Aug	60[a]	60[ae]	60[a]	38	63[c]
	Sep	68[a]	46[bf]	68[a]	46	71[c]
	Oct	52	53	53	31[de]	56
	Nov	39	39	39	42	42[d]
	Dec	51[c]	28[def]	28[de]	30	30[de]
1988	Jan	40	41	41	20[e]	43
	Feb	30	31	31	33	34
	Mar	21	21	22	24	24
	Apr	35	36	36	38	39[c]
	May	26	26	27	28	30
	Jun	39	39	40	18[de]	43[c]
	Jul	50[ae]	51[ae]	51[ae]	29	31[d]
	Aug	59[a]	60[ae]	60[a]	38	63[a]
	Sep	66	66[a]	67	45	70
	Oct	72[a]	73[ce]	73[a]	51	76
	Nov	55	56	56	34[c]	58
	Dec	40	40	40	43	43

[a] There is a *very low* probability (less than one percent) that the moon may be sighted 24 hours earlier if the conditions are *extremely* good.

[b] There is a *very low* probability (less than one percent) that the moon may be sighted 24 hours later if the conditions are *extremely* poor.

[c] The site is inside the zone of uncertainty and the moon may perhaps be sighted 24 hours later.

[d] The site is inside the zone of uncertainty and the moon may perhaps be first sighted 24 later.

[e] This prediction disagrees with the prediction of Ilyas.

[f] This prediction disagrees with the prediction of the U.S. Naval Observatory.